

# Scale interactions and scaling laws in rotating flows at moderate Rossby numbers and large Reynolds numbers

P.D. Mininni<sup>1,2</sup>, A. Alexakis<sup>3</sup>, and A. Pouquet<sup>2</sup>

<sup>1</sup> *Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, 1428 Buenos Aires, Argentina.*

<sup>2</sup> *NCAR, P.O. Box 3000, Boulder, Colorado 80307-3000, U.S.A.*

<sup>3</sup> *Laboratoire Cassiopée, Observatoire de la Côte d'Azur, BP 4229, Nice Cedex 04, France.*

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The effect of rotation is considered to become important when the Rossby number is sufficiently small, as is the case in many geophysical and astrophysical flows. Here we present direct numerical simulations to study the effect of rotation in flows with moderate Rossby numbers (down to  $Ro \approx 0.1$ ) but at Reynolds numbers large enough to observe the beginning of a turbulent scaling at scales smaller than the energy injection scale. We use coherent forcing at intermediate scales, leaving enough room in the spectral space for an inverse cascade of energy to also develop. We analyze the spectral behavior of the simulations, the shell-to-shell energy transfer, scaling laws, and intermittency, as well as the geometry of the structures in the flow. At late times, the direct transfer of energy at small scales is mediated by interactions with the largest scale in the system, the energy containing eddies with  $k_{\perp} \approx 1$ , where  $\perp$  refers to wavevectors perpendicular the axis of rotation. The transfer between modes with wavevector parallel to the rotation is strongly quenched. The inverse cascade of energy at scales larger than the energy injection scale is non-local, and energy is transferred directly from small scales to the largest available scale. Also, as time evolves and the energy piles up at the large scales, the intermittency of the direct cascade of energy is preserved while corrections due to intermittency are found to be the same (within error bars) as in homogeneous turbulence.

## I. INTRODUCTION

Strong rotation is present in many geophysical and astrophysical flows. Its effect is considered to become important when the Rossby number (the ratio of the convective to the Coriolis acceleration, or the ratio of the rotation period to the eddy turn over time) is sufficiently small. The large scales of atmospheric and oceanic flows for example are affected by the rotation of the Earth. The Rossby number for mid-latitude synoptic scales in the atmosphere is  $Ro \approx 0.1$  [37]. In the Sun, the typical Rossby number in the convective zone is  $Ro \approx 0.1 - 1$  [30]. Furthermore, the Reynolds number ( $Re$ , the ratio of the convective to the viscous acceleration) in these systems is also very large, and the flows are in a turbulent state.

Besides rotation, stratification is also important in the atmosphere, the ocean, and other geophysical flows. Many studies have considered solely the effect of rotation in a turbulent flow, as a first step to gain better understanding of the fluid dynamics of geophysical systems. For rapid rotation (very small Rossby numbers), significant progress has been made by applying resonant wave theory [7, 20, 23, 43] and weak turbulence theory [22]. In these approaches, the flow is considered as a superposition of inertial waves with a short period, and the evolution of the system for long times is derived considering the effect of resonant triad interactions. This explains successfully the observed enhanced transfer of energy from the small to the large scales [40], and sheds light on the mechanism that drives the flow to be quasi-two dimensional at large scales [41, 43]. Energy in three dimen-

sional modes is transferred by a subset of the resonant interactions to modes with smaller vertical wavenumber. Spectral closures [14, 15] give similar results.

However, resonant wave theories are only valid when the rotation period is much shorter than the eddy turnover time at all scales. For large Reynolds numbers, small scales are excited with a characteristic timescale proportional to the eddy turnover time, that decreases as the scales become smaller. Therefore the approximations made in such theories can break down at sufficiently small scales, provided that the Reynolds number is large enough for these scales to be excited. How the results of resonant wave theory extend to the case of only moderate Rossby numbers but very large Reynolds numbers is still unclear. Several phenomenological theories have been developed to consider the case with large  $Re$  (see e.g. [16, 36, 45, 46]) leading to different results for the scaling of the energy spectrum.

In numerical simulations, the study of rotating turbulent flows is constrained by the computational cost of properly resolving the inertial waves and the resonant triadic interactions, together with the cost of resolving the small scale fluctuations when  $Re$  is large. Inverse cascades were shown to develop and anisotropies to appear in low resolution ( $32^3$  and  $64^3$  grid points) simulations [8, 9, 24], either solving the equations of motion directly or using a subgrid model. Small aspect ratio boxes were considered in [39, 41] allowing for an increase in resolution. Simulations at higher resolution were done later by [44] studying in particular the behavior of the shell-to-shell energy transfer. Recently, simulations with large Reynolds number and small Rossby number were performed using  $128^3$  grids and 8th-order hyperviscosity

[18], thus confirming the dominant role of resonant triads for rapid rotation at large  $Re$ , although the results also suggest that resonant wave theories can be valid only for a finite interval of time. All these simulations also give different results for the scaling of the energy spectrum at scales larger than the forcing scale; it was shown in [40], using a truncated model, that this can be the result of how all the relevant timescales are resolved.

Here we study the effect of rotation in a turbulent flow in high resolution direct numerical simulations (up to  $512^3$  grid points). Simulations at this resolution were also performed recently in [36]; in this case, energy was injected at the largest scale available and the focus was solely on the scaling of small-scale fluctuations, showing depletion of the energy cascade and reduced intermittency. Our main objective, on the other hand, is to study the statistical properties of the fluctuations in flows with moderate Rossby numbers (down to  $Ro \approx 0.1$ ) but at Reynolds numbers large enough to observe the beginning of a turbulent scaling at scales smaller than the energy injection scale. To this end, we use coherent forcing at intermediate scales, leaving enough room in the spectral space for an inverse cascade of energy to develop when the Rossby number is small enough. We also use the largest value of the Reynolds number allowed by our grid to observe a direct transfer of energy at small scales. After describing the simulations, we study its spectral behavior, the shell-to-shell energy transfer, scaling laws and intermittency, and finally the geometry of the structures in the flow.

## II. NUMERICAL SIMULATIONS

We solve numerically the equations for an incompressible rotating fluid with constant mass density,

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega} \times \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla \mathcal{P} + \nu \nabla^2 \mathbf{u} + \mathbf{F}, \quad (1)$$

and

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

where  $\mathbf{u}$  is the velocity field,  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$  is the vorticity,  $\mathcal{P}$  is the total pressure (modified by the centrifugal term) divided by the mass density, and  $\nu$  is the kinematic viscosity. Here,  $\mathbf{F}$  is an external force that drives the turbulence, and we chose the rotation axis to be in the  $z$  direction:  $\boldsymbol{\Omega} = \Omega \hat{z}$ , with  $\Omega$  the rotation frequency.

The mechanical forcing  $\mathbf{F}$  is given by the Taylor-Green (TG) flow [42]

$$\mathbf{F} = F_0 [\sin(k_0 x) \cos(k_0 y) \cos(k_0 z) \hat{x} - \cos(k_0 x) \sin(k_0 y) \cos(k_0 z) \hat{y}], \quad (3)$$

where  $F_0$  is the forcing amplitude. Although the forcing injects energy directly only into the  $x$  and  $y$  components of the flow, in the absence of rotation ( $\Omega = 0$ ) the

TABLE I: Parameters used in the simulations.  $N$  is the linear grid resolution,  $k_0$  the wavenumber used in the forcing,  $\nu$  the kinematic viscosity,  $\Omega$  the rotation rate,  $t_{\max}$  the maximum number of turnover times computed;  $Re$ ,  $Ro$ , and  $Ek$  are respectively the Reynolds, Rossby and Ekman numbers.

Run	$N$	$k_0$	$\nu$	$\Omega$	$t_{\max}$	$Re$	$Ro$	$Ek$
A1	256	2	$2 \times 10^{-3}$	0.08	45	900	4.50	$5 \times 10^{-3}$
A2	256	2	$2 \times 10^{-3}$	0.40	45	900	0.70	$8 \times 10^{-4}$
A3	256	2	$2 \times 10^{-3}$	0.80	45	900	0.35	$4 \times 10^{-4}$
A4	256	2	$2 \times 10^{-3}$	1.60	45	900	0.17	$2 \times 10^{-4}$
A5	256	2	$2 \times 10^{-3}$	3.20	150	900	0.09	$1 \times 10^{-4}$
A6	256	2	$2 \times 10^{-3}$	8.00	185	900	0.03	$3 \times 10^{-5}$
B1	512	4	$8 \times 10^{-4}$	0.40	17	1100	1.40	$1 \times 10^{-3}$
B2	512	4	$8 \times 10^{-4}$	1.60	25	1100	0.35	$3 \times 10^{-4}$
B3	512	4	$8 \times 10^{-4}$	8.00	40	1100	0.07	$6 \times 10^{-5}$

resulting flow is fully three-dimensional because of pressure gradients that excite the  $z$  component of the velocity [35, 42]. The resulting flow has no net helicity, although locally regions with strong positive and negative helicity develop. It is also worth noting that this forcing injects zero energy in the  $k_z = 0$  modes, whose amplification observed in the strongly rotating cases is only due to a cascade process.

Two sets of runs were done at resolutions of  $256^3$  (set A) and  $512^3$  grid points (set B). The parameters for all the runs are listed in Table I. With Taylor-Green forcing, the spherical shell in Fourier space where energy is injected has wavenumber  $k_F = \sqrt{3}k_0$ , or equivalently, at a scale  $L_F = 2\pi/k_F$ . For the runs in set A,  $k_F \approx 3.5$ , and for the runs in set B,  $k_F \approx 6.9$ ; as a result, there is more room in spectral space for an inverse cascade to take place in the B runs.

All the runs in set A were started from a fluid at rest. At  $t = 0$ , the rotation and the external forcing were switched on, until reaching a turbulent steady state, or until an inverse cascade was well developed in the case of large rotation rates. The runs in set B were done as follows. Run B1 was started from a fluid at rest and after turning on the rotation and external forcing, the run was continued to reach a turbulent steady state. Runs B2 and B3 were started from a snapshot of the velocity field from the steady state of run B1, and both runs were continued until a new steady state was reached, or an inverse cascade developed. This latter method proved useful in saving computing time, as no differences were observed when comparing the late time evolution of the runs in the two sets. In all simulations, a dissipative range was properly resolved, and the time step was much smaller than all the relevant timescales.

We define the integral and Taylor scales of the flow respectively as

$$L = 2\pi \frac{\int E(k) k^{-1} dk}{\int E(k) dk}, \quad (4)$$

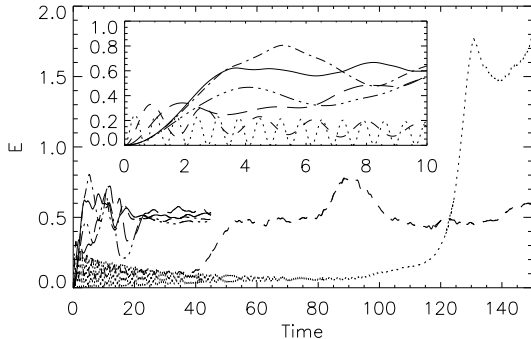


FIG. 1: Time history of the energy for set A: A1 (solid), A2 (dash-dot), A3 (dash-triple dot), A4 (long dash), A5 (dash), and A6 (dot); the inset shows a detail of the evolution at early times when waves prevail. Note the large increase in energy as  $Ro$  decreases.

and

$$\lambda = 2\pi \left( \frac{\int E(k) dk}{\int E(k) k^2 dk} \right)^{1/2}, \quad (5)$$

where  $E(k)$  is the energy spectrum. Since for large  $\Omega$  an inverse cascade develops, these two scales are useful to describe the evolution of characteristic scales in the flow with time. However, to avoid a time dependence of the Reynolds and Rossby numbers, we define for each run the Reynolds number as

$$Re = \frac{L_F U}{\nu}, \quad (6)$$

and the Rossby number as

$$Ro = \frac{U}{2\Omega L_F}. \quad (7)$$

We also define the Ekman number as

$$Ek = \frac{Ro}{Re} = \frac{\nu}{2\Omega L_F^2}. \quad (8)$$

The turnover time at the forcing scale is then defined as  $T = L_F/U$  where  $U = \langle u^2 \rangle$  is the r.m.s. velocity measured in the turbulent steady state, or when the inverse cascade starts. The amplitude of the forcing  $F_0$  in the simulations is increased as  $\Omega$  is increased in order to have  $U \approx 1$  in all the runs.

### III. TIME EVOLUTION AND SPECTRA

Figure 1 shows the time history of the energy in the runs in set A. Runs A1-A4 show a similar evolution, but runs A5 and A6 evolve differently. As the Rossby number decreases, a transient develops in which the total energy oscillates with a frequency that increases with  $\Omega$ .

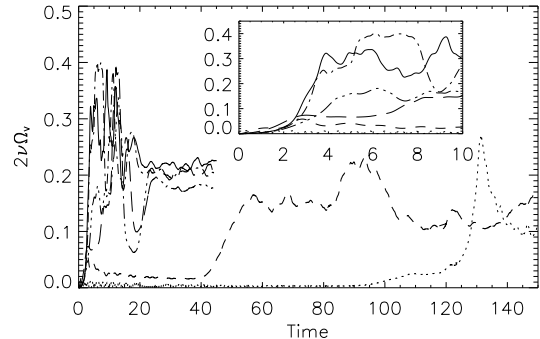


FIG. 2: Time history of the energy dissipation rate (labels as in Fig. 1) the inset shows the evolution at early times.

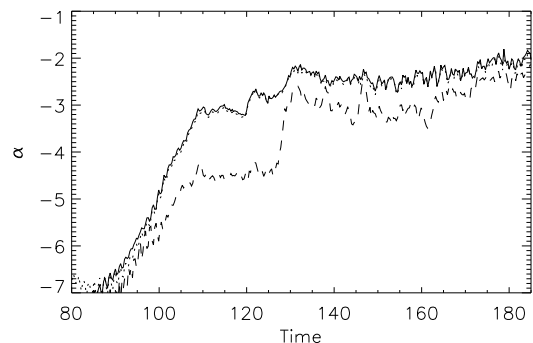


FIG. 3: Spectral index  $\alpha$  as a function of time in run A6, in the isotropic energy spectrum  $E(k)$  (solid), in the  $E(k_\perp)$  spectrum (dot), and in the  $E(k_\parallel)$  spectrum (dash). Note that the energy is dominated by the orthogonal modes.

It takes a longer time for this transient to decay as  $\Omega$  increases, and then the energy increases suddenly and a turbulent regime develops. An inverse cascade of energy is observed in run A6 after  $t \approx 120$ . The increase in the energy observed after this time is also accompanied by a monotonous increase with time of the flow integral scale  $L$ . Even in the runs in set B, that are restarted from a pre-existing turbulent steady state, long runs are needed to reach another turbulent state after turning on the rotation. As an example, in run B3 it takes  $\approx 20$  turnover times for the transient to decay and for an inverse cascade of energy to develop.

The energy dissipation rate  $2\nu \int \omega^2/2dV$  as a function of time is shown in Fig. 2. As the Rossby number decreases, the peak of the dissipation rate is reached at later times, and then it saturates. Note that during the early transient in runs A5 and A6, the dissipation is almost negligible, while in the saturated state the mean dissipation rate decreases slowly with decreasing Rossby number.

The shape of the energy spectrum evolves with time, specially after the transient as turbulence sets in, and

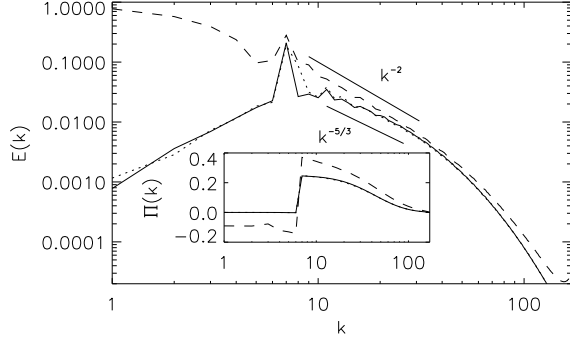


FIG. 4: Isotropic energy spectra at late times in runs B1 (solid,  $t \approx 16$ ), B2 (dot,  $t \approx 24$ ), and B3 (dash,  $t \approx 40$ ) at low  $Ro$ . Two slopes are given as a reference. The inset shows the isotropic energy flux for the same runs.

later again as the spectrum becomes dominated by the contribution from the largest scales when the Rossby number is small enough for an inverse cascade to develop. Figure 3 shows the time evolution of the spectral index  $\alpha$  (the exponent in the region of the spectrum with  $k > k_F$  that follows a power law  $\sim k^\alpha$ ) in run A6. Three curves are shown, which correspond respectively to the spectral index computed in the isotropic energy spectrum  $E(k)$ , in the perpendicular energy spectrum  $E(k_\perp)$  (where  $k_\perp$  denotes the wavevectors perpendicular to  $\Omega$ ), and the parallel spectrum  $E(k_\parallel)$  (where  $k_\parallel$  denotes the wavevectors parallel to  $\Omega$ ).

Before  $t \approx 80$ , we cannot recognize a power law in the energy spectra. After  $t \approx 80$ , the spectral indices in  $E(k)$  and  $E(k_\perp)$  grow monotonically from a value of  $-7$  until reaching a plateau with  $\alpha \approx -3$  at  $t \approx 110$ . The energy spectra  $E(k)$ ,  $E(k_\perp)$ , and  $E(k_\parallel)$  show wide and steep power law behavior from  $t \approx 80$  to  $t \approx 120$ . During this transient, the energy flux is almost zero, as can also be expected from the small value of the energy dissipation in run A6 before  $t \approx 110$  (Fig. 2). The end of the transient at  $t \approx 110$  and the plateau in  $\alpha$  correspond respectively to the increase in the energy and in the energy dissipation rate showed in Figs. 1 and 2. The spectral index in  $E(k_\parallel)$  also has a plateau with  $\alpha \approx -4.5$ . However, as the inverse cascade sets in and the energy piles up at the largest available scale in the system, the spectral index changes again and seem to slowly evolve towards  $\alpha \approx -2$  in both  $E(k)$  and  $E(k_\perp)$ .

Note that the inverse cascade only starts after  $\approx 10$  turnover times after the turbulent state is reached at  $t \approx 110$ . This can be understood as follows. The energy spectrum observed before  $t \approx 110$  has almost no flux. Nonlinear transfer of energy is required for the flow to become two-dimensional under the effect of rotation [15, 41, 43], and the nonlinear transfer is negligible until  $t \approx 110$ . Then, after a few turnover times, the flow undergoes a transition and the inverse cascade sets in.

The early transient is only observed in the runs in set

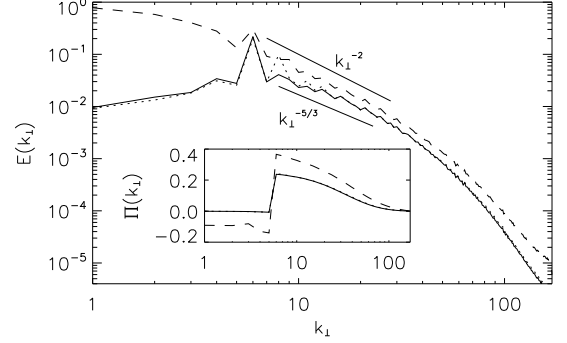


FIG. 5:  $E(k_\perp)$  and  $\Pi(k_\perp)$  (inset) at late times in runs B1, B2, and B3. Labels are as in Fig. 4.

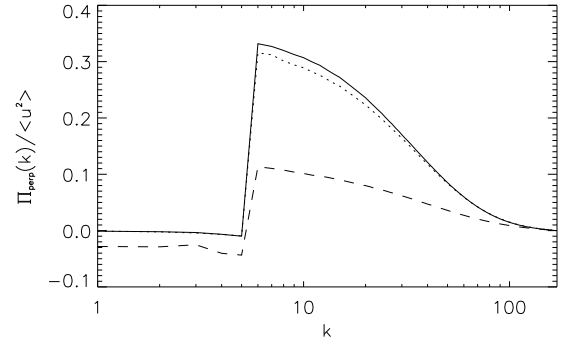


FIG. 6:  $\Pi(k_\perp)/\langle u^2 \rangle$  at late times in runs B1, B2, and B3. Labels are as in Fig. 4.

A, since the runs in set B are started from a turbulent steady state. However, after the transient the spectral evolution of the runs in set A and B is similar. Since runs in set B have more scale separation for an inverse cascade to develop when  $Ro$  is small enough, we focus now on this set of runs. We show in Fig. 4 the isotropic energy spectrum at late times in runs B1-B3. While runs B1 and B2 show no growth of energy at scales larger than the mechanical forcing, except for some backscattering with a  $\sim k^2$  spectrum, run B3 at late times is dominated by the energy in the  $k = 1$  shell. At scales smaller than the forcing scale, the spectrum of run B3 is steeper than that of runs B1 and B2, and compatible with a  $\sim k^{-2}$  scaling. The inset in Fig. 4 shows the isotropic energy flux in the same runs. Note that in run B3, the flux at scales larger than the forcing scale is negative and approximately constant, indicating the development of an inverse cascade of energy for small  $Ro$ . At smaller scales, the energy flux is positive. We thus conclude that in rotating flows, both the direct and inverse energy cascades can cohabit.

The energy spectrum  $E(k_\perp)$  is shown in Fig. 5, together with the energy flux  $\Pi(k_\perp)$ . The spectrum and flux are similar to the isotropic ones (indicating most of

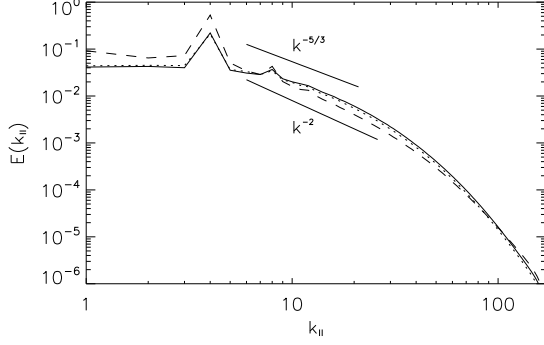


FIG. 7:  $E(k_{\parallel})$  at late times in runs B1, B2, and B3. Labels are as in Fig. 4.

the energy is in these modes), and  $\Pi(k_{\perp})$  confirms the development of an inverse cascade of energy in  $k_{\perp}$  at scales larger than the forcing scale in run B3, and a direct cascade at smaller scales with a  $\sim k_{\perp}^{-2}$  scaling. Figure 6 shows the energy flux  $\Pi(k_{\perp})$  normalized by the r.m.s. velocity in each run. Note that the increase of the flux observed in the inset of Fig. 5 is only due to the increase in the energy of the system as the inverse cascade piles up energy at the largest available scale. As Fig. 6 indicates, the actual transfer of energy is slowed down by the rotation, and run B3 shows a smaller normalized flux than the other two runs at scales smaller than the forcing scale.

On the other hand, there is no clear scaling in the small scales in  $E(k_{\parallel})$ , nor an inverse cascade at large scales (see Figure 7). The  $E(k_{\parallel})$  spectrum in run B3 is steeper than the  $E(k_{\perp})$  spectrum, consistent with the results shown in Fig. 3 for run A6 at late times. Slopes  $\sim k^{-5/3}$  and  $\sim k^{-2}$  are shown in Fig. 7 only as a reference.

#### IV. ENERGY TRANSFER

In this section we study the scale interactions and energy transfer in rotating turbulent flows. A study of the energy transfer in this context, albeit at lower resolution, was done before by [44]. We will focus on runs B1, B2, and B3, that have enough scale separation for direct and inverse cascades to develop when  $Ro$  is small enough. Similar results were obtained in the analysis of the runs in set A.

To investigate the transfer of energy among different scales we consider the shell filter decomposition of the velocity field,

$$\mathbf{u}(\mathbf{x}) = \sum_K \tilde{\mathbf{u}}_K(\mathbf{x}), \quad (9)$$

where  $K$  denotes a foliation of Fourier space in shells,

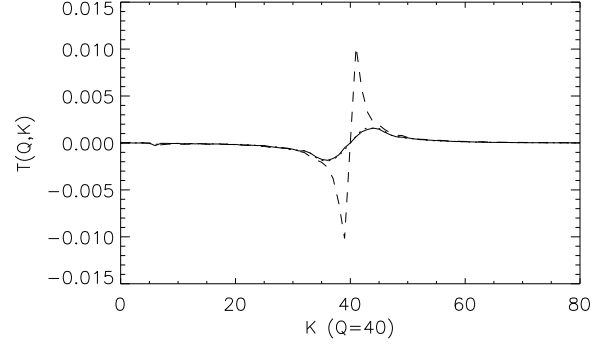


FIG. 8: Shell-to-shell transfer function  $T(Q, K)$  at  $Q = 40$  for runs B1 (solid), B2 (dot), and B3 (dash) at late times.

that for our purposes can be taken as spheres [2, 3, 31, 32]

$$\mathbf{u}_K(\mathbf{x}) = \sum_{K \leq |\mathbf{k}| \leq K+1} \tilde{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (10)$$

cylinders [1]

$$\mathbf{u}_{K_{\perp}}(\mathbf{x}) = \sum_{K \leq |\mathbf{k}_{\perp}| \leq K+1} \tilde{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (11)$$

or planes [1]

$$\mathbf{u}_{K_{\parallel}}(\mathbf{x}) = \sum_{K \leq |\mathbf{k}_{\parallel}| \leq K+1} \tilde{\mathbf{u}}_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}. \quad (12)$$

Then, we can define the shell-to-shell transfer between these shells as

$$T(Q, K) = - \int \mathbf{u}_K(\mathbf{u} \cdot \nabla) \mathbf{u}_Q d\mathbf{x}^3. \quad (13)$$

This function expresses the transfer rate of energy lying in the shell  $Q$  to energy lying in the shell  $K$ . It satisfies the symmetry property  $T(Q, K) = -T(K, Q)$  [3], and the numbers labeling the shells  $Q$  and  $K$  can correspond to any of the foliations of Fourier space listed above [1]. In particular, we will study the cases  $T(Q, K)$ ,  $T(Q_{\perp}, K_{\perp})$ , and  $T(Q_{\parallel}, K_{\parallel})$ . The energy fluxes discussed in the previous section can be reobtained in terms of the shell-to-shell transfer function as

$$\Pi(k) = - \sum_{K=0}^k \sum_Q T(Q, K), \quad (14)$$

where again the wavenumbers  $k$ ,  $K$ , and  $Q$  can correspond to different foliations of Fourier space depending on the subindex.

Note that for the definition of the shells a linear binning is used. Alternatively, the shells can be defined by a logarithmic binning of spectral space with intervals  $(\gamma^n K_0, \gamma^{n+1} K_0]$  for some positive  $\gamma > 1$  and for integer  $n$ .



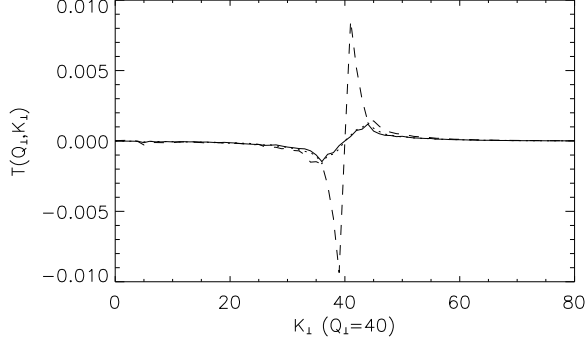


FIG. 9: Shell-to-shell transfer function  $T(Q_{\perp}, K_{\perp})$  at  $Q_{\perp} = 40$  for runs B1, B2, and B3. Labels are as in Fig. 8.

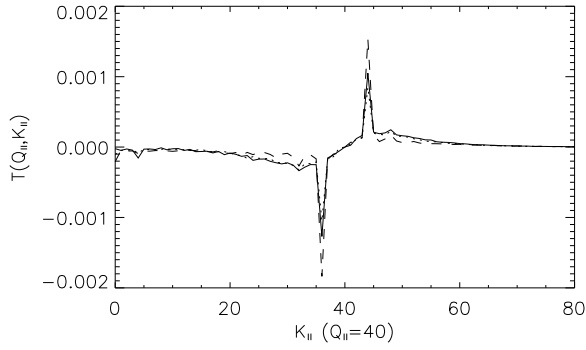


FIG. 10: Shell-to-shell transfer function  $T(Q_{\parallel}, K_{\parallel})$  at  $Q_{\parallel} = 40$  for runs B1, B2, and B3. Labels are as in Fig. 8. Notice these transfers are roughly 5 times weaker than in the  $\perp$  case.

However, logarithmic binning cannot distinguish transfer between linearly spaced neighbor shells (from the shell  $K$  to the shell  $K+1$ ) from the transfer between logarithmic neighbor shells (from  $K$  to  $\gamma K$ ). If the cascade is the result of interactions with the large-scale flow (e.g., with modes with wavenumber  $k_F$  associated to the external forcing), the energy in a shell  $K$  will be transferred to the shell  $K + k_F$ . Logarithmic binning does not distinguish this transfer from the transfer due to local triadic interactions that transfer the energy from  $K$  to  $\gamma K$ . For this reason we use linear binning, but we note that care needs to be taken when using the word “scale” that implies in general a logarithmic division of the spectral space. The transfer among logarithmic shells can be reconstructed at any time later by summing over the linearly spaced shells.

Figure 8 shows the shell-to-shell transfer  $T(Q, K)$  at  $K = 40$  for runs B1, B2, and B3 at late times. The negative peak to the left indicates energy is transferred from these  $K$ -shells to the shell  $Q = 40$ , while the positive peak to the right indicates energy goes from the  $Q = 40$  shell to those  $K$ -shells. In runs B1 and B2 the shell-to-shell transfer peaks at  $|Q - K| \approx k_F$ . This was observed

before in simulations of homogeneous turbulence [2, 32], and indicates that the energy transfer is local (the energy goes from a shell  $Q$  to a nearby shell  $K$ , although the step in the energy cascade is independent of that scale and related to the forcing scale). However, in run B3 the transfer strongly peaks at  $|Q - K| \approx 1$ . The same effect is observed in  $T(Q_{\perp}, K_{\perp})$  shown in Fig. 9. This indicates that at late times in run B3, the direct transfer of energy at small scales is mediated by interactions with the largest scale in the system, the energy containing eddies with  $k_{\perp} \approx 1$  (see Fig. 5). As a result, the timescale associated with the direct cascade of energy in  $k_{\perp}$  increases (and its flux reduces, see Fig. 6), since the energy is transferred in smaller steps in Fourier space than in the case of the B1 and B2 runs.

The shell-to-shell transfer  $T(Q_{\parallel}, K_{\parallel})$  at  $Q_{\parallel} = 40$  for the same runs is shown in Fig. 10. The dependence with the Rossby number of this transfer function is less drastic. In all runs, the transfer function  $T(Q_{\parallel}, K_{\parallel})$  peaks at  $|Q_{\parallel} - K_{\parallel}| \approx k_F$ . Since there is no inverse cascade of energy in  $k_{\parallel}$ , the energy containing scale in this direction does not change as the Rossby number is decreased, and neither does the position of the peaks in  $T(Q_{\parallel}, K_{\parallel})$ . However, note the drop in the amplitude of the transfer in run B3 for all shells except the ones satisfying  $|Q_{\parallel} - K_{\parallel}| = k_F$ . As a result, for small Rossby number the transfer of energy between shells with  $Q_{\parallel}$  and  $K_{\parallel}$  is quenched except for the direct interactions with the external forcing. Most of the interactions responsible for the transfer of energy to small scales between different  $k_{\parallel}$  shells are then interactions with the forcing.

Figure 11 shows the transfer functions  $T(Q_{\perp}, K_{\perp})$  and  $T(Q_{\parallel}, K_{\parallel})$  in runs B1 and B3 for all values of  $K$  and  $Q$  up to 40. In all cases, the white and black bands near  $K \approx k_F$  and  $Q \approx k_F$  indicate a small amount of energy injected by the external forcing that is directly transferred to all wavenumbers up to  $\approx 30$ . For  $K$  and  $Q$  larger than  $k_F$ , the figures confirm the results of the direct cascade of energy presented in Figs. 9 and 10. For wavevectors perpendicular to  $\Omega$ , as the Rossby number is decreased the peaks in  $T(Q_{\perp}, K_{\perp})$  move closer to the diagonal  $K_{\perp} = Q_{\perp}$  [Figs. 11(a) and (b)], indicating the direct cascade in the perpendicular direction takes place in smaller  $k$ -steps given by the largest scale of the system. For all wavevectors, the energy in the parallel direction [see  $T(Q_{\parallel}, K_{\parallel})$  in Figs. 11(c) and (d)] is transferred to smaller scales, and the cascade step does not depend on the Rossby number. However, all transfer except the transfer with  $|Q_{\parallel} - K_{\parallel}| = k_F$  is strongly quenched in run B3.

The development of a non-local inverse transfer can be observed in Fig. 11(b) for  $K_{\perp} < k_f$  and  $Q_{\perp} < k_f$ . The transfer is inverse, since below the diagonal  $Q_{\perp} = K_{\perp}$  regions with negative (dark gray and black)  $T(Q_{\perp}, K_{\perp})$  can be observed. This means that energy is taken from e.g.,  $K_{\perp} = 20$  and transferred to shells with  $Q_{\perp} < k_F$ . The transfer is also non-local, since this inverse transference takes place between disparate scales. The non-

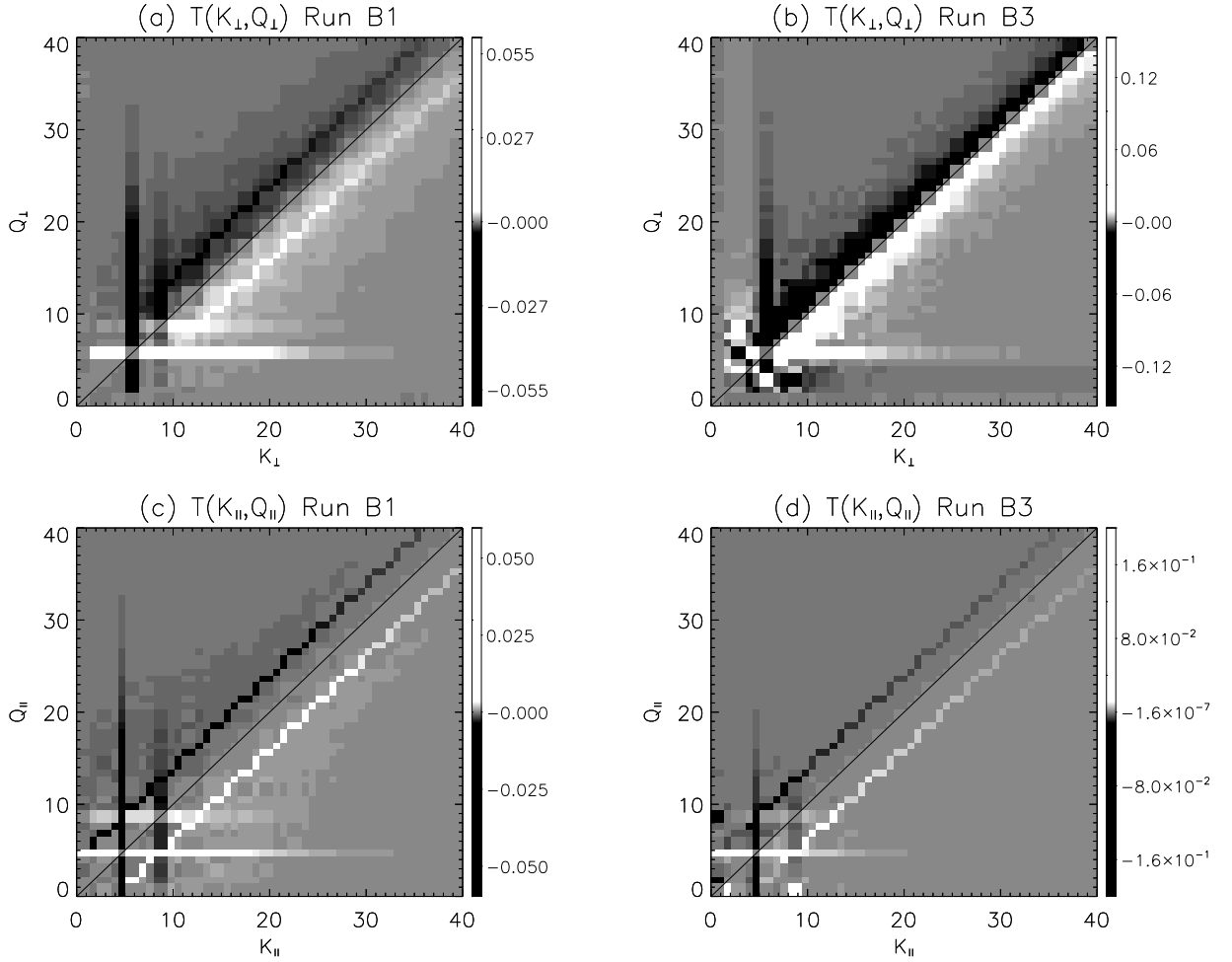


FIG. 11: Shell-to-shell energy transfer functions  $T(Q_{\perp}, K_{\perp})$  (a,b) and  $T(Q_{\parallel}, K_{\parallel})$  (c,d) at late times in runs B1 (a,c) and B3 (b,d). Notice the quenching of the transfer in case (d), except for the interactions with the forcing scale.

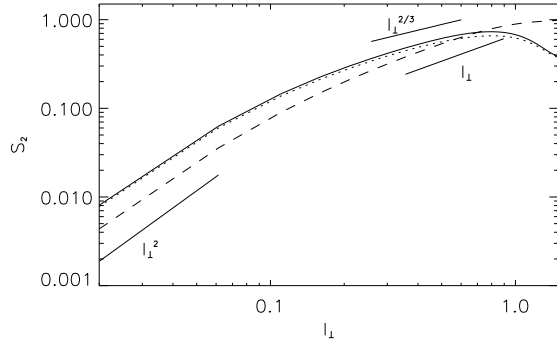


FIG. 12: Second order longitudinal structure function  $S_2(\ell_{\perp})$  (where  $\ell_{\perp}$  denotes increments were taken in the direction perpendicular to  $\Omega$ ) for runs B1 (solid), B2 (dot), and B3 (dash).

local transfer of energy in rotating turbulence shares similarities with the inverse cascade of magnetic helicity in

magnetohydrodynamics (MHD) [4, 5]. Near the diagonal  $Q_{\perp} = K_{\perp}$  the transfer is more complex. The inverse transfer superposes with a (smaller in net amplitude) direct local transfer (dark spots below and near the diagonal, and light spots above and near it, for  $K_{\perp}$  and  $Q_{\perp}$  smaller than  $k_F$ ). This small direct transfer of energy at large scales is the result of a reflection of energy at  $K = 1$ , and was also observed in studies of the inverse cascade of magnetic helicity in MHD [4]. The reflection of energy in Fourier space when it reaches the largest scale in the box suggests that the late time evolution can be dependent on the boundary conditions, a property that was already observed in simulations of two dimensional turbulence [6, 10, 13, 28, 29]. In our case, the simulations do not contain a large-scale dissipation mechanism (such as a hypo-viscosity), and therefore energy piles up at the largest available scale until its growth is stopped by the (small-scale) dissipation.

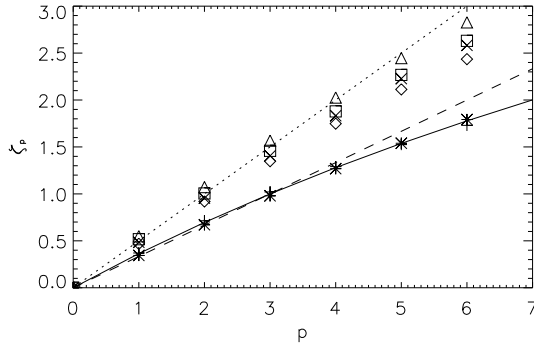


FIG. 13: Scaling exponents  $\zeta_p$  for the steady state of runs B1 (+), B2 (\*), and for run B3 at different times:  $t \approx 20$  ( $\diamond$ ),  $t \approx 25$  ( $\times$ ),  $t \approx 30$  ( $\square$ ), and  $t \approx 40$  ( $\triangle$ ). The solid line corresponds to the scaling exponents given by the She-L       model [38], the dash line is the Kolmogorov prediction  $\zeta_p = p/3$ , and the dotted line is  $\zeta_p = p/2$ .

## V. SCALING LAWS AND INTERMITTENCY

In this section, we consider the anisotropic inertial range scaling of the runs in Table I as described by the longitudinal velocity increments in the direction perpendicular to rotation,

$$\delta u(\mathbf{x}, \ell_\perp) = \hat{\mathbf{r}} \cdot [\mathbf{u}(\mathbf{x} + \ell \hat{\mathbf{r}}) - \mathbf{u}(\mathbf{x})], \quad (15)$$

where  $\hat{\mathbf{r}}$  is a unit vector perpendicular to  $\boldsymbol{\Omega}$ . The longitudinal structure functions  $S_p(\ell_\perp)$  (with displacements along  $\ell_\perp$ ) can then be defined as

$$S_p(\ell_\perp) = \langle \delta u(\mathbf{x}, \ell_\perp)^p \rangle, \quad (16)$$

where the brackets denote spatial average. If the flow is self-similar, we expect  $S_3(\ell_\perp) \sim \ell_\perp^{\zeta_p}$ , where  $\zeta_p$  are the scaling exponents. In isotropic and homogeneous hydrodynamic turbulence, the K      -Howarth theorem implies  $S_3(\ell) \sim \ell$ , and the Kolmogorov energy spectrum follows from the assumption  $S_3(\ell) \sim \ell^{p/3}$  [21]. In practice, the spontaneous development of strong gradients in the small scales of a turbulent flow gives rise to corrections to this scaling, a phenomenon referred to as intermittency.

From dimensional analysis, if the energy spectrum at small scales in rotating turbulence is  $E \sim k_\perp^{-2}$ , we expect  $S_2 \sim \ell_\perp$ . Figure 12 shows the second order structure function for runs B1, B2, and B3 at late times outside the wave regime when the turbulence has developed. At small scales for all runs,  $S_2 \sim \ell_\perp^2$ , consistent with a smooth field in the dissipative range. At large scales,  $S_2$  is larger for run B3 than for runs B1 and B2, a signature of the inverse cascade of energy and of the development of large scale structures in the flow. The scaling of runs B1 and B2 at intermediate scales is compatible with the Kolmogorov spectrum, while the scaling in run B3 is consistent with the  $\sim k_\perp^{-2}$  energy spectrum. Note that such a scaling can be understood as a slow-down in the energy

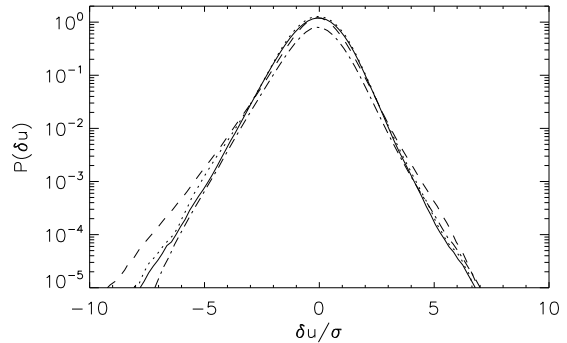


FIG. 14: Pdf of the longitudinal velocity increments ( $\ell_\perp = 3\eta$ ) for run B3 at different times:  $t \approx 20$  (solid),  $t \approx 25$  (dot),  $t \approx 30$  (dash), and  $t \approx 40$  (dash-dot);  $\eta$  is the Kolmogorov dissipative length.

TABLE II: Characteristic scales and dimensionless numbers of the runs in set B.  $t$  is the time,  $L_\parallel$  and  $L_\perp$  are the integral scales using respectively the  $E(k_\parallel)$  and  $E(k_\perp)$  spectra,  $\lambda$  is the isotropic Taylor scale,  $Ro_\lambda$  is the micro-Rossby number, and  $\mu = 2\zeta_3 - \zeta_6$ .

Run	$t$	$L_\parallel$	$L_\perp$	$\lambda$	$Ro_\lambda$	$\mu$
B1	16	1.5	0.9	0.29	3.70	$0.23 \pm 0.01$
B2	24	0.9	1.6	0.31	0.91	$0.24 \pm 0.01$
B3	20	2.6	1.2	0.50	0.12	$0.19 \pm 0.02$
B3	25	2.4	1.5	0.55	0.11	$0.26 \pm 0.02$
B3	30	2.1	1.7	0.59	0.12	$0.26 \pm 0.05$
B3	40	1.9	2.8	0.53	0.33	$0.24 \pm 0.02$

transfer rate because of interactions between waves and eddies (see e.g., [34, 36, 46]); such a slow-down is consistent with the results of the transfer function presented in the previous section. Considering the energy flux in the inertial range  $\epsilon$  is slowed down by (see e.g., [25, 26])

$$\epsilon \sim \delta u_{\ell_\perp}^2 \tau_\Omega / \tau_{\ell_\perp}^2, \quad (17)$$

where  $\tau_\Omega \sim 1/\Omega$ , and  $\tau_{\ell_\perp} \sim \ell_\perp / \delta u_{\ell_\perp}$  is the turnover time of eddies in the plane perpendicular to  $\boldsymbol{\Omega}$ ; the scaling

$$\delta u_{\ell_\perp}^2 \sim \ell_\perp \quad (18)$$

follows.

Figure 13 shows the scaling exponents  $\zeta_p$  up to order 6 computed in runs B1, B2, and B3. The scaling exponents are defined as the exponents in

$$S_p(\ell_\perp) \sim \ell_\perp^{\zeta_p} \quad (19)$$

in the inertial range associated to the direct cascade of energy (i.e., for  $\ell_\perp < L_F$ ). Runs B1 and B2 behave as non-rotating turbulence, with Kolmogorov scaling ( $\zeta_p \approx 2/3$ ) and intermittency corrections (the prediction  $\zeta_p = p/3$  of Kolmogorov, and the intermittency model of intermittency in homogeneous and isotropic turbulence of She



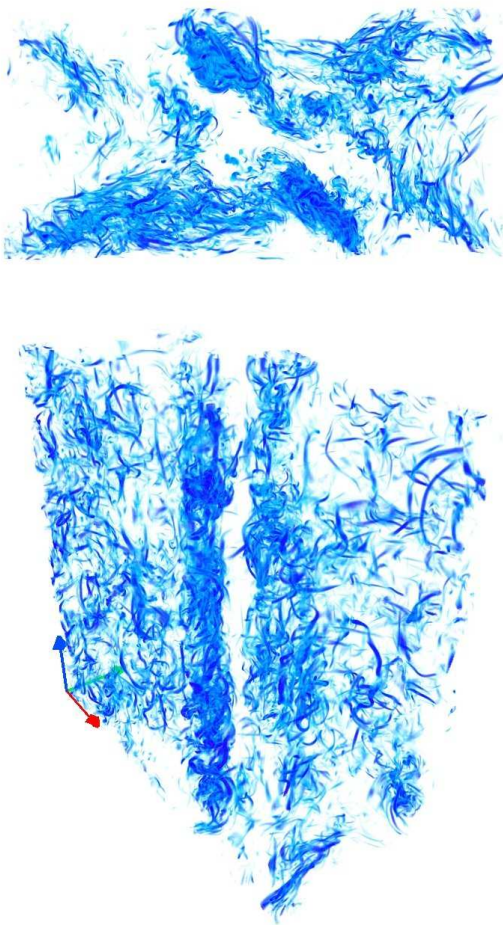


FIG. 15: (Color online) Three dimensional rendering of the vorticity intensity in a subvolume of  $256 \times 512 \times 512$  grid points of run B2. The top view shows the subvolume in the direction of the axis of rotation; in the bottom view the red and blue arrows indicate respectively the  $x$  and  $z$  axis.

and L       [38] are shown in Fig. 13 as a reference). However, run B3 has a distinct behavior with  $\zeta_2 \approx 1$ . As time evolves in this run, and the energy piles up at  $k_\perp \approx 1$ , the second order scaling exponent slowly converges to this value. Low order moments follow the curve  $\zeta_p = p/2$ , but high order moments deviate from the straight line. The level of intermittency in the flow in all these runs can be measured in terms of  $\mu = 2\zeta_3 - \zeta_6$ . This quantity, together with the integral scales of the flow (based on the parallel and perpendicular energy spectra), the Taylor scale, and the micro-Rossby number (based on the Taylor scale of the flow),

$$Ro_\lambda = \frac{U}{2\Omega\lambda}, \quad (20)$$

are given in Table II for the runs in set B at different times.

It can be seen that at late times run B3 evolves towards an anisotropic state in the large scales, with

$L_\perp/L_\parallel \approx 1.5$ . However, at small scales the flow seems more isotropic and at late times ( $t \approx 40$ ) in this run  $\lambda_\perp/\lambda_\parallel \approx 0.8$ . The micro-Rossby number in runs B1, B2, and B3 take different values in the range 0.11–3.7. However, the value of  $\mu$  is, within error bars, approximately the same for all the runs. As a result, the intermittency in the direct cascade of energy seems to be independent of the Rossby number  $Ro$  and the micro-Rossby number  $Ro_\lambda$ .

Finally, Figure 14 shows the time evolution of the probability density function (pdf) of the longitudinal velocity increments in run B3. Increments in the direction perpendicular to  $\Omega$  were computed, and the increment was taken equal to three times the Kolmogorov dissipation scale  $\eta$  in each run. The velocity increments in each run were normalized by their corresponding root mean square deviation  $\sigma$ . In agreement with the level of intermittency observed in the scaling exponents, the pdfs show exponential tails indicating a larger than Gaussian probability of large gradients to occur in the small scales. The amplitude of the tails of the pdfs as a function of  $\delta u/\sigma$  does not change significantly with time. Moreover, the root mean square deviation  $\sigma$  of the velocity increments  $\delta u$  increases with time. So if the pdfs are plotted versus  $\delta u$  (instead of versus  $\delta u/\sigma$ ), the pdfs actually become wider at later times. This effect can be understood considering that once the inverse cascade of energy sets in, the total energy in the flow as a function of time increases.

## VI. STRUCTURES

The intermittency reported in the previous section in the scaling exponents and the pdfs of velocity increments indicates that even after the inverse cascade sets in, the flow develops strong velocity gradients in the small scales. In this section, we present visualizations of the flow and consider the structures that emerge.

Figure 15 shows a three dimensional rendering of the vorticity intensity in half of the computational domain ( $256 \times 512 \times 512$  grid points) at late times. The top view corresponds to the subvolume in the direction of the axis of rotation. Only regions with strong vorticity are shown. Note that the flow is anisotropic and quasi-2D, as it is clear from the top view. In the bottom view, the development in the flow of large scale column-like structures can be seen. However, the columns display small scale structures with thin vortex filaments. These filaments seem to be ordered according to the large scale pattern. The presence of regions with strong vorticity even when the Rossby number is small enough for the inverse cascade of energy to develop can be expected from the results shown in Figs. 13 and 14, linked to the intermittency of the flow.

The local relative helicity  $\omega \cdot \mathbf{u}/(|\omega||\mathbf{u}|)$  in the same subvolume is shown in Fig. 16. Unlike in isotropic and homogeneous turbulence, regions of strong vorticity are not correlated with regions of strong relative helicity. The

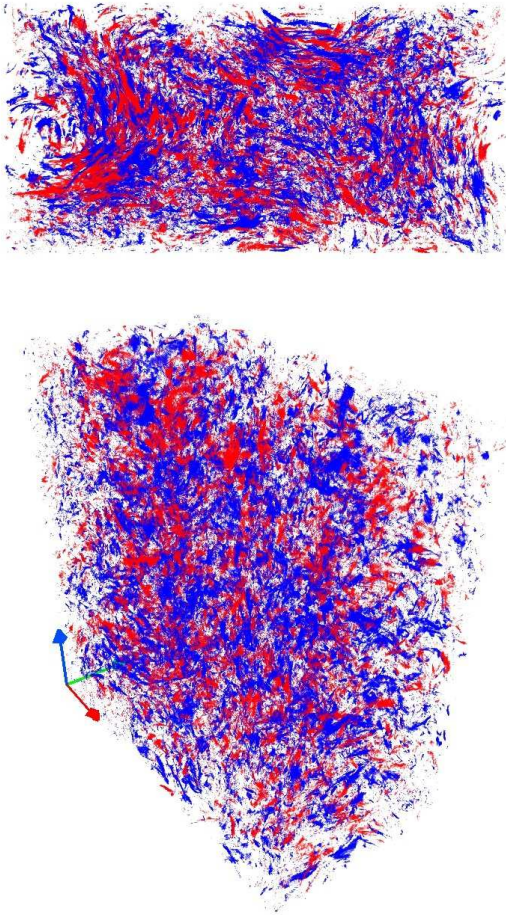


FIG. 16: (Color online) Three dimensional rendering of the relative helicity intensity in a subvolume of  $256 \times 512 \times 512$  grid points of run B2. Blue corresponds to positive helicity, and red to regions with negative helicity. Only regions with  $|\boldsymbol{\omega} \cdot \mathbf{u}|/(|\boldsymbol{\omega}||\mathbf{u}|) > 0.95$  are shown, indicating strong quenching of nonlinearities. Viewing points are identical to Fig. 15.

net helicity over the entire box averages to zero, and locally regions of positive and negative helicity seem to be more isotropic and homogeneous than the other quantities studied. Note that the local relative helicity can be quite ubiquitously strong, indicative of a local quenching of nonlinear interactions.

## VII. CONCLUSIONS

In this work, we presented results of the study of the turbulent scaling laws and energy transfer in direct numerical simulations of rotating flows in periodic domains. Spatial resolutions of  $256^3$  (set A) and of  $512^3$  grid points (set B) were used, while moderate Rossby numbers (down to  $Ro \approx 0.1$ ) and large Reynolds numbers (up to  $Re \approx 1100$ ) were considered, with enough scale separation to observe both a direct and an inverse

cascade of energy when the rotation was strong enough. Runs in set A were started from a fluid at rest, while runs in set B were restarted from a previous state of homogeneous turbulence. In the former case, for  $Ro \approx 0.1$ , a long transient was found in which the energy dissipation is small, as well as the energy flux to smaller scales. During this transient, the energy spectrum has a wide but steep spectrum, and its slope monotonously increases as a function of time. After turbulence sets in and the inverse cascade of energy develops, the energy spectrum evolves towards a  $E \sim k_{\perp}^{-2}$  scaling at scales smaller than the forcing scale. This late time evolution is observed in both sets of runs.

At late times, the energy flux in runs A5, A6, and B3 indicates an inverse cascade of energy in  $k_{\perp}$  at scales larger than the forcing scale, and a direct cascade of energy at smaller scales. The net flux to small scales decreases as the Rossby number decreases, while the amplitude of the flux to large scales increases. No inverse cascade is observed in  $k_{\parallel}$ . These cascades were confirmed by the study of the shell-to-shell energy transfer. The direct transfer of energy at scales smaller than the forcing is local, although in the runs with small Rossby number the transfer in  $k_{\perp}$  is significantly slowed down. In this direction, the energy is transferred between shells  $K_{\perp}$  and  $Q_{\perp}$  with small steps given by  $|Q_{\perp} - K_{\perp}| \approx 1$ . As a result, the direct transfer of energy in  $k_{\perp}$  at small scales is mediated by interactions with the largest scale in the system, the energy containing eddies with  $k_{\perp} \approx 1$ . The timescale associated to the direct cascade in  $k_{\perp}$  then increases, and its flux reduces. In  $k_{\parallel}$  the transfer is direct at all scales, and a larger component than in the case of non-rotating turbulence is due to interactions with the forcing scale. These results are in good agreement with phenomenological derivations of the energy spectrum in rotating turbulence that consider a slow down in the energy transfer rate because of interactions between waves and eddies [36, 46]. The non-local interactions also lead to the development of anisotropies in the flow [43].

The inverse cascade of energy that develops at scales larger than the forcing scale in runs A5, A6, and B3 is non-local, in the sense that the transfer of energy associated to this cascade takes place between disparate shells in Fourier space. At late times, the inverse transfer superposes with a (smaller in amplitude) direct local transfer of energy. This small direct transfer of energy at large scales is the result of a reflection at  $k_{\perp} = 1$ , when the peak of energy reaches the largest scale in the box. Consequently, the late time evolution of simulations of rotating turbulence may depend on the boundary conditions used, a property already observed in simulations of two dimensional turbulence [6, 10, 13, 28, 29].

The study of structure functions in the direct cascade range shows that the second order scaling exponent for increments perpendicular to the rotation in runs with small  $Ro$  is  $\zeta_2 \approx 1$ , in agreement with the energy spectrum. Low order moments follow the curve  $\zeta_p = p/2$  but high order moments deviate from this law, an indication

of intermittency. The level of intermittency in the direct cascade of energy, as measured by the exponent  $\mu$ , is the same for runs with and without rotation. The spontaneous formation of strong gradients in the small scales is further confirmed by pdfs of the velocity increments and by visualization of regions of strong vorticity in the flow.

More separation of scales is needed to study the intermittency in the inverse cascade of energy. Because of its relation to small scale gradients, intermittency is believed to be associated only with the forward cascade of energy. The intermittency phenomenon is not observed in the velocity field in two dimensional turbulence for which the conservation of vorticity leads to an inverse energy cascade to the large scales [11, 12], although intermittency in the vorticity (which cascades directly to small scales) is observed. It is unclear how the dual cascade of energy (towards small and large scales) in rotating turbulence affects the intermittency in the inverse cascade range. While intermittency is associated with small scale events, in many cases the strong events can affect the dynamics of the large scales, specially in systems close to

criticality; as an example, intermittency is a possible explanation for the occurrence of extended minima in solar activity [17, 33]; it is also known to affect the transport of momentum in atmospheric surface layers [27].

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